Isotropy groups of locally finitely presentable and extensive categories

Dedicated to Pieter Hofstra

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Introduction

- Pieter and his collaborators introduced the topos-theoretic phenomenon of *isotropy* in the 2012 paper
  - [2] Jonathon Funk, Pieter Hofstra, and Benjamin Steinberg, *Isotropy and crossed toposes*, Theory Appl. Categ. **26** (2012), No. 24, 660–709.
- They showed that every Grothendieck topos $\mathcal{E}$ has a canonical internal group object called its *isotropy group*, which acts canonically on every object of $\mathcal{E}$ and formally generalizes the notion of conjugation. (It is the group object representing a certain functor $\mathcal{Z}_{\mathcal{E}} : \mathcal{E}^{\mathsf{op}} \to \mathbf{Grp}$.)
- When I started my PhD at uOttawa with Pieter and Phil Scott, Pieter suggested that I try to provide more explicit characterizations of the isotropy groups of certain toposes; namely, the *classifying toposes* of algebraic and essentially algebraic theories.

Introduction

- Every geometric theory $\mathbb{T}$ (in particular, every essentially algebraic theory) has a *classifying topos* $\mathcal{B}(\mathbb{T})$, which is a cocomplete topos that has a *universal model* of $\mathbb{T}$ and classifies all topos-theoretic models of $\mathbb{T}$.

- When $\mathbb{T}$ is essentially algebraic, the classifying topos of $\mathbb{T}$ is the presheaf topos $\mathbf{Set}^{\mathbb{T}\text{-}\mathbf{Mod_{fp}}}$, where $\mathbb{T}$-$\mathbf{Mod_{fp}}$ is the (essentially small) category of *finitely presented* models of $\mathbb{T}$. So the isotropy group of the topos $\mathbf{Set}^{\mathbb{T}\text{-}\mathbf{Mod_{fp}}}$, which we call the *isotropy group of* $\mathbb{T}$, is a certain group object $\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Mod_{fp}} \to \mathbf{Grp}$.

- Pieter and Phil essentially suggested that I try to explicitly characterize $\mathcal{Z}_{\mathbb{T}}$ in terms of the syntax and semantics of $\mathbb{T}$, and Pieter referred me to a paper by George Bergman [1] that treated the case where $\mathbb{T}$ is the algebraic theory of groups.

Motivation

- In this paper [1], Bergman proved that the inner automorphisms of groups can be characterized purely *categorically* (without mentioning conjugation), as the automorphisms that extend naturally along every group homomorphism.

- To see this, observe first that if α is an inner automorphism of a group G (induced by $s \in G$), then for each group morphism $f : G \to H$ with domain G, we can 'push forward' α to define an inner automorphism

$$\alpha_f : H \xrightarrow{\sim} H$$

by conjugation with $f(s) \in H$ (so that $\alpha_{\mathbf{id}_G} = \alpha$).

Motivation

The resulting family of automorphisms $(\alpha_f)_f$ is *natural/coherent*, in the sense that it defines a natural automorphism of $G/\mathbf{Grp} \to \mathbf{Grp}$. So let us call an *arbitrary* natural automorphism

$$\left(\alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f)\right)_{\mathbf{dom}(f)=G}$$

of $G/\mathbf{Grp} \to \mathbf{Grp}$ an *extended inner automorphism* of G.

Theorem (Bergman [1])

Let G be a group and let $\alpha : G \xrightarrow{\sim} G$ be an automorphism of G. Then α is an **inner** *automorphism of G iff there is an extended inner automorphism $(\alpha_f)_f$ of G with $\alpha = \alpha_{\mathbf{id}_G}$.*

This provides a completely *categorical* characterization of inner automorphisms of groups: they are exactly those group automorphisms that are 'coherently extendible' along morphisms out of their domain.

Covariant isotropy

- We obtain a functor $\mathcal{Z}_{\mathbf{Grp}} : \mathbf{Grp} \to \mathbf{Grp}$ that sends every group G to its group of extended inner automorphisms. (Bergman's theorem entails that $\mathcal{Z}_{\mathbf{Grp}} \cong 1 : \mathbf{Grp} \to \mathbf{Grp}$.)

- In fact, *every* category $\mathbb{C}$ has a *covariant isotropy group (functor)*

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \to \mathbf{Grp}$$

that sends each object C of $\mathbb{C}$ to its group of extended inner automorphisms, i.e. natural automorphisms

$$\left(\alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f)\right)_{\mathbf{dom}(f)=C}$$

of the projection functor $C/\mathbb{C} \to \mathbb{C}$.

Covariant isotropy

- We can also turn Bergman's characterization of inner automorphisms in **Grp** into a *definition* of inner automorphisms in an arbitrary category $\mathbb{C}$: we say that an automorphism $\alpha : C \xrightarrow{\sim} C$ of an object C of $\mathbb{C}$ is *inner* if there is an extended inner automorphism $(\alpha_f)_f \in \mathcal{Z}_{\mathbb{C}}(C)$ with $\alpha_{\mathbf{id}_C} = \alpha$.

- When $\mathbb{C} = \mathbb{T}\text{-}\mathbf{Mod_{fp}}$ for an essentially algebraic theory $\mathbb{T}$, we have $\mathcal{Z}_{\mathbb{T}\text{-}\mathbf{Mod_{fp}}} = \mathcal{Z}_{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Mod_{fp}} \to \mathbf{Grp}$. In other words, the isotropy group of (the classifying topos of) $\mathbb{T}$ is precisely the covariant isotropy group of the category $\mathbb{T}\text{-}\mathbf{Mod_{fp}}$.

The isotropy group of an essentially algebraic theory

- We generalized ideas from the proof of Bergman's Theorem to give a logical/syntactic characterization of the isotropy group $\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Mod}_{\mathbf{fp}} \to \mathbf{Grp}$ of an essentially algebraic theory $\mathbb{T}$ (which we think of as being presented as a *quasi-equational partial Horn theory* in the sense of Palmgren-Vickers [7]), and more generally of the covariant isotropy group $\mathcal{Z}_{\mathbb{T}\text{-}\mathbf{Mod}} : \mathbb{T}\text{-}\mathbf{Mod} \to \mathbf{Grp}$ of the category $\mathbb{T}$-**Mod**.

- Using the syntax of $\mathbb{T}$, we can define a notion of *definable automorphism* for a model M of $\mathbb{T}$, and the definable automorphisms of each $M \in \mathbb{T}\text{-}\mathbf{Mod}$ form a group $\mathbf{DefInn}(M)$.

Definable automorphisms

- Specifically, if $\mathbb{T}$ is single-sorted, then given $M \in \mathbb{T}$-**Mod**, one can form the $\mathbb{T}$-model $M\langle \mathbf{x} \rangle$ obtained from M by freely adjoining an indeterminate element $\mathbf{x}$. Elements of $M\langle \mathbf{x} \rangle$ are congruence classes $[t]$ of terms t involving $\mathbf{x}$ and constants from M, where two terms s, t are congruent if they are provably equal in the *diagram theory* of $M\langle \mathbf{x} \rangle$.
- We define $\mathbf{DefInn}(M)$ to be the group of all elements $[t] \in M\langle \mathbf{x} \rangle$ that provably define automorphisms of the $\mathbb{T}$-model M.
- If $\mathbb{T}$ is multi-sorted, one can extend the above definitions appropriately.

The isotropy group of an essentially algebraic theory

We then proved:

Theorem ([5])

*Let $\mathbb{T}$ be an essentially algebraic theory. For each $M \in \mathbb{T}$-**Mod**, the group $\mathcal{Z}_{\mathbb{T}\text{-}\mathbf{Mod}}(M)$ of extended inner automorphisms of M is isomorphic to the group $\mathbf{DefInn}(M)$ of definable automorphisms of M (naturally in $M \in \mathbb{T}$-**Mod**).*

In particular, an automorphism $\alpha : M \xrightarrow{\sim} M$ is *inner* iff there is some definable automorphism in $\mathbf{DefInn}(M)$ that *induces* α.

Initial examples [4]

- If $\mathbb{T}$ is the theory of sets, then $\mathbb{T}$ has trivial isotropy group, i.e. $\mathcal{Z}_{\mathbf{Set}}(S) \cong \mathbf{DefInn}(S)$ is the trivial group for each set S.

- If $\mathbb{T}$ is the theory of groups, then Bergman proved for each group G that

$$\mathcal{Z}_{\mathbf{Grp}}(G) \cong \mathbf{DefInn}(G) \cong \left\{ \left[g\mathbf{x}g^{-1} \right] \in G\langle \mathbf{x} \rangle \ \mid g \in G \right\} \cong G.$$

- If $\mathbb{T}$ is the theory of monoids, then for each monoid M we have

$$\mathcal{Z}_{\mathbf{Mon}}(M) \cong \mathbf{DefInn}(M) \cong \left\{ \left[m\mathbf{x}m^{-1} \right] \in M\langle \mathbf{x} \rangle \mid m \in \mathbf{Inv}(M) \right\}.$$

Initial examples [4]

- If $\mathbb{T}$ is the theory of abelian groups, then for each abelian group G we have
$$\mathcal{Z}_{\mathbf{Ab}}(G) \cong \mathbf{DefInn}(G) \cong \{[\mathbf{x}], [-\mathbf{x}]\} \cong \mathbb{Z}_2.$$

- If $\mathbb{T}$ is the theory of commutative monoids or unital rings, then $\mathbb{T}$ has trivial isotropy group.

- If $\mathbb{T}$ is the theory of (not necessarily commutative) unital rings, then for every such ring R we have
$$\mathcal{Z}_{\mathbb{T}}(R) \cong \mathbf{DefInn}(R) \cong \left\{ \left[r\mathbf{x}r^{-1} \right] \in R\langle \mathbf{x} \rangle \mid r \in \mathbf{Unit}(R) \right\} \cong \mathbf{Unit}(R).$$

- If $\mathbb{T}$ is the theory of categories or groupoids, then $\mathbb{T}$ has trivial isotropy group.

Strict monoidal categories

- Let $\mathbb{T}$ be the theory of strict monoidal categories. We proved in [5] that for each (small) strict monoidal category $\mathbb{C}$, the group $\mathbf{DefInn}(\mathbb{C})$ consists of exactly the monoidal *inner* automorphisms, i.e. the automorphisms $F : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ for which there is some $\otimes$-invertible object $c \in \mathbf{ob}\,\mathbb{C}$ such that F is given by *conjugation* with c, i.e.

$$(a \in \mathbb{C}) \qquad\qquad F(a) = c \otimes a \otimes c^{-1}.$$

- We then deduced that

$$\mathcal{Z}_{\mathbb{T}}(\mathbb{C}) \cong \mathbf{DefInn}(\mathbb{C}) \cong \mathbf{Inv}\left(\mathbf{Ob}(\mathbb{C})\right),$$

the group of $\otimes$-invertible elements of the object monoid of $\mathbb{C}$, also known as the *Picard group* of $\mathbb{C}$. Pieter and Martti Karvonen then extended this work to arbitrary monoidal categories in [3].

Presheaf categories

- In [5] we also characterized the covariant isotropy group of every *presheaf category*. Given a small category $\mathcal{J}$, we can define an essentially algebraic theory $\mathbb{T}^{\mathcal{J}}$ such that $\mathbb{T}^{\mathcal{J}}$-$\mathbf{Mod} \cong \mathbf{Set}^{\mathcal{J}}$.

- For each presheaf $F : \mathcal{J} \to \mathbf{Set}$, we showed that $\mathbf{DefInn}(F)$ consists of exactly the natural automorphisms $\alpha : F \xrightarrow{\sim} F$ induced by some element $\psi \in \mathbf{Aut}\,(1_{\mathcal{J}})$, in the sense that

$$(k \in \mathcal{J}) \qquad \alpha_k = F(\psi_k) : F(k) \xrightarrow{\sim} F(k).$$

- It follows that the covariant isotropy group $\mathcal{Z}_{\mathbf{Set}^{\mathcal{J}}} : \mathbf{Set}^{\mathcal{J}} \to \mathbf{Grp}$ is *constant* on the group $\mathbf{Aut}\,(1_{\mathcal{J}})$ of natural automorphisms of $1_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$.

Extensive categories

In subsequent work [9] I showed that the result for presheaf categories essentially extends to every *(finitely) extensive* category (including every elementary topos). Specifically, we have the following result:

Theorem

Let $\mathbb{C}$ be a finitely extensive category with initial object 0. Then $\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \to \mathbf{Grp}$ is constant on $\mathcal{Z}_{\mathbb{C}}(0) \cong \mathbf{Aut}\,(1_{\mathbb{C}})$.

Conclusions and further work

- Given any essentially algebraic theory $\mathbb{T}$, we provided an explicit logical description of the isotropy group $\mathcal{Z}_{\mathbb{T}\text{-}\mathbf{Mod_{fp}}} = \mathcal{Z}_{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Mod_{fp}} \to \mathbf{Grp}$ of $\mathbb{T}$ in terms of definable automorphisms. This provides a complete characterization of the covariant isotropy groups of all locally finitely presentable categories. I have also characterized the covariant isotropy group of every extensive category.

- My preprint [8] (currently under review) characterizes the covariant isotropy group of many categories of the form $\mathbb{T}\text{-}\mathbf{Mod}^{\mathcal{J}}$ (for small $\mathcal{J}$ and essentially algebraic $\mathbb{T}$). In [3], Pieter and Martti developed the theory of 2-categorical isotropy. Pieter's last MSc student, Frédéric LeBlanc, recently finished his thesis [6] on the isotropy *Lawvere theory* of an essentially algebraic theory.

Thank you!

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