Introduction To Locally Posetal and Linear Bicategories

Richard Blute
University Of Ottawa

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Fundamental groupoids and homotopies

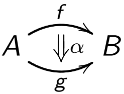
- Given a space, one builds a category where the objects are points, arrows are homotopy equivalence classes of paths, and composition is path composition. This is called the *fundamental groupoid*. (It's a groupoid because every arrow is invertible. Just take the opposite of the path.)
- For path connected spaces, this is equivalent to the fundamental group.
- But modding out by an equivalence relation erases information.
- Can we build a structure which keeps track of homotopies?
- It won't be a category, since we are no longer modding out by reparametrizations.

Fundamental groupoids and homotopies, continued

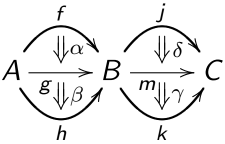
- One can talk about homotopies between homotopies, and homotopies between homotopies between homotopies, etc.. so we imagine a structure:
  - 0-cells: objects
  - 1-cells: arrows between 0-cells
  - 2-cells: arrows between 1-cells
  - ...
  - k-cells: arrows between k-1-cells
  - ...
- Organizing all this information into a coherent definition, ∞-*categories*, and then working with it, is a huge ongoing project.
- *Grothendieck's homotopy hypothesis*: spaces and ∞-groupoids are the same thing.

Horizontal and vertical structure

It's helpful to think of the one-cells as being horizontal and two cells vertical:



Then we have vertical and horizontal composition:



Bicategories

- A *bicategory* $\mathcal{B}$ (Jean Benabou) consists of a class of objects $A, B, C, \ldots$, and for each pair of objects (0-cells), a category $\mathcal{B}(A, B)$. The objects of $\mathcal{B}(A, B)$ are called 1-cells and the arrows 2-cells.
- We have composition functors $\mathcal{B}(A, B) \times \mathcal{B}(B, C) \dot{\to} \mathcal{B}(A, C)$ and identity objects $I_A \in \mathcal{B}(A, A)$.
- Composition is associative up to a specified isomorphism and identities must act as identities up to a specified isomophism.
- These isomorphisms must satisfy coherence equations. The coherence theorem for bicategories was proved by Mac Lane and Paré. It's a generalization of Mac Lane's coherence theorem for monoidal categories.
- Coherence for *tricategories* was worked out by Gordon, Power and Street. It's an 81 page paper.

Sidenote on terminology

- There is a distinction between *bicategories* and *2-categories*.
- 2-categories are bicategories for which composition is associative up to equality, not just isomorphism, and similarly for identities.
- Cat, whose 0-cells are (small) categories, 1-cells are functors and natural transformations are 2-cells, is a 2-category.
- But some authors, especially John Baez and the *nLab* community, call bicategories 2-categories and 2-categories *strict* 2-categories.

Limits and colimits in bicategories

For more details on limits and colimits in bicategories, see article on nLab. We'll just look at one simple case.

Definition

Let $\mathcal{B}$ be a bicategory and X, Y 0-cells in $\mathcal{B}$. A product of X and Y consists of a 0-cell, denoted $X \times Y$, and two 1-cells:

$$\pi_1 \colon X \times Y \to X \qquad \pi_2 \colon X \times Y \to Y$$

such that for all 0-cells A, the functor

$$\mathcal{B}(A, X \times Y) \to \mathcal{B}(A, X) \times \mathcal{B}(A, Y)$$

is an equivalence of categories.

There are also various types of 2-limits that are not just lifted versions of 1-limits.

Rel

- Consider the (ordinary) category whose objects are sets and arrows are binary relations, i.e. $R\colon X \to Y$ is $R \subseteq X \times Y$.
- Composition is defined as: Given $R\colon X \to Y$ and $S\colon Y \to Z$, then:

$$R;S\colon X \to Z \quad \text{is} \quad (x,z) \in R;S \text{ iff } \exists y \in Y(x,y) \in R \& (y,z) \in S$$

- We'll call this category Rel. It's monoidal, in fact $*$-autonomous, in fact compact closed. It's also a Seely model, i.e. a full model of all of linear logic.
- But we're going to ignore all of that structure, for now.

Rel, continued

- Instead we'll focus on the fact that $\text{Rel}(X, Y) = \mathcal{P}(X \times Y)$. This is a partially ordered set under inclusion, and hence a category. So this is a fairly straightforward case of having arrows between arrows. 2-cells are inclusions.
- Furthermore, composition preserves the order in each variable.
- So, Rel is a *locally posetal bicategory*, i.e. one for which all of the Hom-categories, $\mathcal{B}(A, B)$, are posets.
- One can ignore coherence issues in this case.
- There is still enough structure to illustrate the key points of bicategories.

Rel, continued continued

- Note that $\mathrm{Rel}(X, Y) = \mathrm{Set}(X \times Y, \mathrm{Bool})$, where $\mathrm{Bool} = \{\mathbf{0}, \mathbf{1}\}$. We're equating a subset with its characteristic function.

- From this point of view, composition can be written as

$$R; S(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z)$$

- What can we replace Bool with to get a generalized notion of relation? The answer will be *quantales*.

- But before getting to the definition, we'll motivate it by looking at a classic paper by Bill Lawvere, *Metric spaces, generalized logic and closed categories*.

- See also the book *Monoidal Topology*, edited by Dirk Hofmann, Gavin J. Seal and Walter Tholen.

Rel, continued continued continued

- Let's first just think about endorelations on a set X. So $a \subseteq X \times X$, or $a\colon X \times X \to \mathrm{Bool}$. One can ask for the relation a to be transitive and reflexive. This amounts to the following inequalities:

$$a(x,y) \wedge a(y,z) \leq a(x,z) \qquad \mathbf{1} \leq a(x,x)$$

  We're viewing $\wedge$ as an associative unital binary operation on Bool. This should remind you of the maps necessary to have a *monad* or a *category*. We'll come back to this.

- Now let's replace Bool with:

$$P^{+} = ([0,\infty]^{op}, +, 0)$$

  Then the above inequalities become

$$a(x,y) + a(y,z) \geq a(x,z) \qquad 0 \geq a(x,x)$$

  Thus a can be thought of as a *(generalized) metric* on X.

Rel, continued continued continued continued

- Let's go back to our definition of Rel and let's replace Bool with P^+.
- Then we get a composition formula for $a\colon X \times Y \to P^+$ and $b\colon Y \times Z \to P^+$, as follows:

$$a;b(x,z) = inf_{y \in Y}(a(x,y) + b(y,z))$$

- This is easily seen to be associative with units for each object. We want to think of a, b as *metric relations.*
- What did we need to make this work? We needed an associative, unital multiplication, and for the poset to have arbitrary sups. But we need one more thing to ensure that the multiplication is associative.

Sup is good food.

Definition

A partially-ordered set which has arbitrary suprema is a *sup-lattice*.

Evidently, a poset with all sups also has all infs, but we call these sup-lattices because we want morphisms that preserve those.

Definition

The category Sup has sup-lattices as objects and sup-preserving maps as arrows.

Theorem

Sup *is a symmetric monoidal closed (in fact* $*$*-autonomous) category.*

The tensor product classifies maps that are sup-preserving in each variable precisely as the tensor product of vector spaces classifies maps that are linear in each variable.

Monadic structure

We have a forgetful functor $U\colon \mathrm{Sup} \to \mathrm{Poset}$, which has a left adjoint F, and Sup will be equivalent to the Eilenberg-Moore category for the corresponding monad.

Definition

- Let X be a poset. A subset $U \subseteq X$ is *downclosed* if $x \in U$ and $y \leq x$, then $y \in U$.
- Let $\mathrm{Dwn}(X) = \{U \subseteq X | U \quad \text{is downclosed}\}$. $\mathrm{Dwn}(X)$ is a poset under inclusion.
- If $x \in X$, define $\downarrow x = \{y | y \leq x\}$. Evidently, $\downarrow x$ is downclosed.
- Define a map $\eta\colon X \to \mathrm{Dwn}(X)$ by $x \mapsto \downarrow x$. Evidently, η is order-preserving.

Monadic structure II

Lemma

- $\mathrm{Dwn}(X)$ *is a sup-lattice for every poset* X.
- *Let* X *be a poset. (Hence both* X *and* $\mathrm{Dwn}(X)$ *are categories.) Then* $\eta\colon X \to \mathrm{Dwn}(X)$ *has a left adjoint if and only if every subset of* X *has a join, i.e. is a sup-lattice.*
- Dwn *determines a monad on* Poset, *for which* η *is the unit.* $\mu_X\colon \mathrm{Dwn}(\mathrm{Dwn}(X)) \to \mathrm{Dwn}(X)$ *is simply the union.*

Theorem

Sup *is equivalent to the Eilenberg-Moore category of the monad* Dwn.

Quantales

Definition (C. Mulvey)

A *quantale* is a monoid in Sup. Equivalently, it is a partially ordered set Q with all suprema and an associative multiplication $\otimes \colon Q \times Q \to Q$ with unit $\top$ such that for all subsets $P \subseteq Q$ and all elements $a \in Q$, we have

$$\left(\bigvee P\right) \otimes a = \bigvee_{p \in P} p \otimes a \quad \text{and} \quad a \otimes \left(\bigvee P\right) = \bigvee_{p \in P} a \otimes p$$

Note that Q necessarily satisfies $a \otimes \mathbf{0} = \mathbf{0} = \mathbf{0} \otimes a$, where $\mathbf{0}$ is the bottom element.

- We'll also assume an identity for the multiplication, which we'll denote by $\top$. Not all authors assume a unit.
- Bool and P^+ are quantales.
- If M is a monoid, $\mathcal{P}(M)$ is a quantale. If $A, B \in \mathcal{P}(M)$, then $A \otimes B = \{ab | a \in A, b \in B\}$.

Examples From Rings

Let R be a ring. The following are examples of quantales. In each case, the multiplication is as follows. If $A, B \subseteq R$. Then

$$A \otimes B = \{a_0 b_0 + a_1 b_1 + \ldots + a_n b_n | a_i \in A, b_i \in B\}$$

- The set of additive subgroups of R
- The set of left ideals of R
- The set of right ideals of R
- The set of 2-sided ideals of R

Similarly, if one has a C^*-algebra, the sets of closed such subsets of the above form quantales.

Locales

See P. Johnstone, *Stone Spaces*.

Definition

A *locale, or frame,* is a quantale for which the associative operation is $\wedge$.

- Locales are the same thing as *complete Heyting algebras*.
- Let X be a topological space. Then $\mathcal{O}(X)$ is a locale. There is a construction in the other direction, assigning a space to each quantale, which is an adjunction. This is *pointless topology*.
- There is a notion of *measurable locale*, and one gets a category which is contravariantly equivalent to the category of commutative von Neumann algebras. This is *pointless measure theory*.

Closed structure

- Every quantale Q is in particular a poset and hence a category.
- The multiplication and unit make it a monoidal category.
- Since the multiplication commutes with sups (colimits), it is a consequence of Freyd's *General Adjoint Functor Theorem* that for all $q \in Q$, the functors $q \otimes (-)$ and $(-) \otimes q$ have right adjoints. We'll denote them $\Leftarrow$ and $\Rightarrow$.
- Actually you don't need GAFT, you can just write down the formulas.
- The adjoints are frequently called *residuations*.

But most importantly, we can now define Q-valued relations.

Q-Rel

If Q is a quantale, we can form the category Q-Rel whose objects are sets and arrows $f\colon X \to Y$ are functions $f\colon X \times Y \to Q$. Given $f\colon X \to Y$ and $g\colon Y \to Z$, composition is defined by

$$f \otimes g(x,z) = \bigvee_{y \in Y} f(x,y) \otimes g(y,z)$$

Note that the use of $\otimes$ on the left refers to composition and on the right refers to multiplication in Q.

Identities are given by

$$id_X(x,x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases}$$

Lemma

If Q is a quantale, then Q-Rel is a locally posetal bicategory, under pointwise order.

Enriched category theory

- It is frequently the case that the Hom-sets of a category have additional structure besides being sets. For example, the set of linear maps between vector spaces is a vector space.
- An *enriched category* generalizes the idea of a category by replacing hom-sets with objects from a general monoidal category.
- If X, Y, Z are objects in a category enriched over a monoidal category $\mathcal{C}$, then $Hom(X, Y), Hom(Y, Z)$ and $Hom(X, Z)$ will be objects of $\mathcal{C}$, and we'll have a composition

$$Hom(X, Y) \otimes Hom(Y, Z) \xrightarrow{;} Hom(X, Z)$$

- Similarly, I need identities, which will now be arrows in $\mathcal{C}$ of the form $I \to Hom(A, A)$.
- This structure is subject to several axioms. See G.M. Kelly-*Basic Concepts of Enriched Category Theory*.

Enriched category theory II

- For example, every symmetric monoidal closed category is enriched over itself. In particular, k-Vec is enriched over itself.
- Similarly, Sup is enriched over itself.
- (Ordinary) categories are enriched over Set.
- 2-categories are Cat-enriched categories.
- A non-example is Top, the category of topological spaces and continuous maps. It is not enriched over itself. A solution to this is to consider Kelley's *compactly generated spaces*, which form a cartesian closed category.
- Enriched category theory is a much larger subject than this. For example, one can consider *enriched bicategories* which are categories enriched over a monoidal bicategory.

Q-Rel II

Definition

A *quantaloid* is a category enriched over Sup. In particular, we note that a one-object quantaloid is the same as a quantale.

The category Rel is a quantaloid, and more generally:

Lemma

Q-Rel *is a quantaloid.*

Exercise: Let $\mathcal{Q}$ be a quantaloid. Construct $\mathcal{Q}$-Rel valued matrices.

Maps

Definition

A 1-cell $f\colon A \to B$ in a locally posetal bicategory is a *map* if there is a morphism $g\colon B \to A$ such that

$$id_A \leq g; f \qquad\qquad f; g \leq id_B$$

In other words, f has a right adjoint.

Exercise

In the locally posetal bicategory Rel, *a relation is a map if and only if it is (the graph of) a function.*

Maps II

Given a quantale Q, one can view a set-function $f\colon X \to Y$ as an arrow in Q-Rel via the formula:

$$f_\circ\colon X \times Y \to Q = \begin{cases} \mathbf{0} & \text{if } y \neq f(x) \\ \top & \text{if } y = f(x) \end{cases}$$

Using this interpretation, functions will always be maps in the sense of the previous slide. But the converse is in general false.

Theorem

Suppose Q has the property that $\top = \mathbf{1}$, the top element, and further satisfies that

$$\forall u, v \in Q, u \vee v = \mathbf{1} \text{ and } u \otimes v = \mathbf{0}, \text{ then } u = \mathbf{1} \text{ or } v = \mathbf{1}$$

then an arrow in Q-Rel is a map if and only if it is a set-theoretic function.

Q-categories

Q-categories are categories enriched over Q, which is itself a monoidal category.

- Let Q be a quantale. An arrow $a\colon X \to X$ in Q-Rel is *transitive* if $a \otimes a \leq a$ and *reflexive* if $id_X \leq a$. A *Q-category* is a set equipped with a transitive, reflexive Q-relation. These conditions amount to:

$$a(x,y) \otimes a(y,z) \leq a(x,z) \qquad \top \leq a(x,x)$$

- If (X,a) and (Y,b) are Q-categories, then a Q-functor $f\colon (X,a) \to (Y,b)$ is a function $f\colon X \to Y$ such that

$$\forall x, x' \in X, a(x,x') \leq b(f(x), f(x'))$$

Examples of Q-categories

- If $Q = \text{Bool}$, then write $x \leq y$ for $a(x, y) = 1$, then a Bool-category is just an ordered set. Q-functors are order-preserving functions.

- If $Q = P^{+}$, then a Q-category is a metric space. Q-functors are non-expansive maps.

- If Q is any quantale, then Q is a Q-category with relation $\multimap$ defined by

$$m \multimap n = \bigvee \{q \in Q | m \otimes v \leq n\}$$

Then Q is a Q-category.

More on Q-categories

Theorem

Let $X = (X, a)$ and $Y = (Y, b)$ be Q-categories. Define $Q - \mathrm{Cat}(X, Y) = \{f : X \to Y | f \text{ is a } Q\text{-functor}\}$.

- *Define a Q-relation on $Q - \mathrm{Cat}(X, Y)$ by:*

$$[f, g] = \bigwedge_{x \in X} b(f(x), g(x))$$

- *Define a Q-relation on $X \times Y$ by*

$$(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$$

If Q is a commutative quantale, then the above are both Q-relations and make Q-Cat a symmetric monoidal closed category.

Modules between ordered sets

The following is a concept from ordered set theory that we'll generalize.

Definition

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be ordered sets. A *module from X to Y* is a relation $R\colon X \to Y$ such that for all $x, x' \in X, y, y' \in Y$:

$$x \leq x', \quad x'Ry, \quad y \leq y' \quad \implies \quad xRy'$$

- This is equivalent to saying that R is a module if and only if the corresponding characteristic function $R\colon X^{op} \times Y \to \text{Bool}$ is monotone.
- Modules between ordered sets can be composed. The resulting category shares many of the properties of Rel.
- See *Monoidal Topology* for more details.

Q-modules, aka Q-profunctors, aka Q-distributors

Let $X = (X, a)$ and $Y = (Y, b)$ be Q-categories.

Definition

A *Q-module* from X to Y is a Q-relation $r \colon X \to Y$ such that

$$a \otimes r \leq r \text{ and } r \otimes b \leq r$$

- Since $\top \leq a(x, x)$, the opposite inequalities also hold. So we have $a \otimes r = r$ and $r \otimes b = r$.
- Modules are closed under composition, so we get a category Q-Mod.
- Q-Mod is a quantaloid, with order inherited from Q-Rel.

Finiteness spaces

- Finiteness spaces were introduced by Ehrhard as a model of full linear logic (i.e. including the necessary monad/comonad structure).
- In previous work (RB, Beauvais-Feisthauer, Cockett, Dewan, Drummond, Jacqmin, Scott) it is shown that finiteness spaces can systematically be used to replace what would be infinitary computations by finite such.
- In fact this idea is already contained in Ehrhard's notion of *linearization*.
- It works again here. If one considers Q-relations over finiteness spaces, the infinite sups and infs needed in quantales are no longer required.

Fniteness spaces I

Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^\perp$ by:

$$\mathcal{U}^\perp = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

- $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \mathcal{V}^\perp \subseteq \mathcal{U}^\perp$
- $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^\perp$

A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$. The elements of $\mathcal{U}$ are called *finitary* subsets.

Finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  1. (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  2. (2) For all $v' \in \mathcal{F}(\mathbb{Y})^\perp$, we have $Rv' \in \mathcal{F}(\mathbb{X})^\perp$.

Composition is relational and it is straightforward to verify that this is a category. We denote it FinRel.

Lemma

In the definition of morphism of finiteness spaces, condition (2) can be replaced with:

$$(2')\ \text{For all } b \in |\mathbb{Y}|, \text{ we have } R\{b\} \in \mathcal{F}(\mathbb{X})^\perp.$$

Finiteness spaces III: It's a model of linear logic

Theorem

FinRel *is a symmetric monoidal closed (in fact, $*$-autonomous) category. The tensor*

$$\mathbb{X} \otimes \mathbb{Y} = (|\mathbb{X} \otimes \mathbb{Y}|, \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}))$$

is given by setting $|\mathbb{X} \otimes \mathbb{Y}| = |\mathbb{X}| \times |\mathbb{Y}|$ *and*

$$\begin{aligned} \mathcal{F}(\mathbb{X} \otimes \mathbb{Y}) &= \{u \times v \mid u \in \mathcal{F}(\mathbb{X}), v \in \mathcal{F}(\mathbb{Y})\}^{\perp\perp} \\ &= \{w \mid \exists u \in \mathcal{F}(\mathbb{X}), \exists v \in \mathcal{F}(\mathbb{Y}), w \subseteq u \times v\}. \end{aligned}$$

We note that it also has sufficient structure to model the rest of the connectives of linear logic.

Linearizing finiteness spaces

Let R be a ring. Let $X = (X, \mathcal{U})$ be a finiteness space. Suppose $f \colon X \to \mathrm{R}$ is a function. Define the *support* of a function to be $supp(f) = \{x \in X | f(x) \neq 0\}$. Then define

$$\mathrm{R}{<}X{>} = \{f \colon X \to \mathrm{R} | supp(f) \in \mathcal{U}\}$$

This is an R-module under obvious pointwise operations. We refer to it as the *linearization* of X. The category Fin(R) will have finiteness spaces as objects and elements of $\mathrm{R}{<}X \multimap Y{>}$ as morphisms.

Now suppose $A \in \mathrm{R}{<}X \multimap Y{>}$ and $B \in \mathrm{R}{<}Y \multimap Z{>}$. Define $AB \in \mathrm{R}{<}X \multimap Z{>}$ by the formula

$$C(x,z) = \sum_{y \in Y} A(x,y)B(y,z)$$

Linearizing II

Lemma

The above formula is well-defined and determines a linear map $A\colon \mathrm{R}{<}X{>}\to \mathrm{R}{<}Z{>}$.

The linearity of the formula is straightforward. The finiteness structure ensures that the above sum is finite. (Exercise.)

Theorem (Ehrhard)

There is a category $\mathrm{Fin}(\mathrm{R})$ *whose objects are finiteness spaces and whose arrows from* X *to* Y *are the elements of* $\mathrm{R}{<}X \multimap Y{>}$. *If* R *is a commutative ring, then* $\mathrm{Fin}(\mathrm{R})$ *is a* $*$*-autonomous category.*

Adding order

To modify the above to be more like $\mathcal{Q}$ -valued relations, we must replace rings with an appropriate ordered structure.

Definition

A *partially ordered semiring* (or *po-semiring*) is a semiring $(\mathsf{R}, +, \cdot, 0, 1)$ together with a partial order on the elements of R, such that

- $x \leq y$ implies $x + z \leq y + z$.
- $x \leq y, 0 \leq z$ imply $xz \leq yz$ and $zx \leq zy$.

A po-semiring is *positive* if every element is greater than or equal to 0.

Adding order II

Positive po-semirings are extremely important in the theory of weighted automata among other places. See:

- M. Droste, W. Kuich. Semirings and Formal Power Series
- Droste, W. Kuich, H. Vogler. Handbook of Weighted Automata

Theorem

If R *is a positive po-semiring, then* Fin(R) *is a locally posetal bicategory, where hom-sets are ordered pointwise.*

$*$-autonomy

Definition

A symmetric monoidal closed category is $*$-*autonomous* if there is an object $\perp$ such that the canonical natural transformation

$$A \longrightarrow (A \Rightarrow \perp) \Rightarrow \perp$$

is a natural isomorphism. We denote $A^{\perp} = A \Rightarrow \perp$.

Examples:

- $k - \mathrm{Vec}_{fd}$
- Rel

We can use a sort of de Morgan duality to define a second associative operation.

$$A \oplus B = (A^{\perp} \otimes B^{\perp})^{\perp}$$

$*$-autonomy II

These correspond to the two multiplicative connectives of linear logic:

$$\text{multiplicative conjunction} = \otimes \qquad \text{multiplicative disjunction} = \oplus$$

Alternatively, one can take the $\otimes$ and $\oplus$ as primitive.

Definition (Cockett-Seely)

A category $\mathcal{C}$ is *linearly distributive* if equipped with two symmetric monoidal structures $(\mathcal{C}, \otimes)$ and $(\mathcal{C}, \oplus)$ related by a linear distribution

$$A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

and symmetric versions of this with similar arrows involving the units and various coherence equations.

$*$-autonomy III

Definition (Cockett-Seely)

A *linearly distributive category with negation* is a linearly distributive category together with a function on objects $A \mapsto A^{\perp}$ and maps:

$$A \otimes A^{\perp} \to \perp \qquad \top \to A^{\perp} \oplus A$$

satisfying coherence equations.

Theorem (Cockett-Seely)

Linearly distributive categories with negation are the same thing as $*$-autonomous categories.

There are noncommutative versions of all of the above.

Linear bicategories [Cockett, Koslowski and Seely]

There is the following standard observation:

Monoidal categories are the same thing as 1-object bicategories.

We'd like to complete the following analogy:

Linearly distributive categories are the same thing as 1-object ??.

The answer will be *linear bicategories*.

Linear bicategories II

As usual with bicategories, one begins with a class of *0-cells* which we will denote $\mathcal{B}_0 = \{X, Y, Z, \ldots\}$. Then for every pair of 0-cells, one has a category $\mathcal{B}(X, Y)$. The objects of $\mathcal{B}(X, Y)$ are called *1-cells* and the arrows are called *2-cells*. But now we have two composition functors which we denote by $\otimes$ and $\oplus$:

$$\otimes, \oplus \colon \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \longrightarrow \mathcal{B}(X, Z)$$

such that each of these compositions gives a bicategory structure. Thus for each composition we have all of the morphisms and coherence that this entails.

Linear bicategories III

These two bicategory structures are related by linear distributions as follows. Given:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow W$$

we have two functors:

$$-\otimes(-\oplus-), (-\otimes-)\oplus- : \mathcal{B}(X, Y)\times\mathcal{B}(Y, Z)\times\mathcal{B}(Z, W)\longrightarrow \mathcal{B}(X, W$$

and we require a natural transformation between them, which is not necessarily an isomorphism:

$$- \otimes(- \oplus -) \Longrightarrow (- \otimes -) \oplus - .$$

There are symmetric versions of this transformation as well as transformations involving the units.

Linear bicategories IV

- Every bicategory is a linear bicategory in an obvious way.
- As a more interesting example, the authors offer Rel with its usual composition, but they also give a second composition. Given $R\colon X \to Y$ and $S\colon Y \to Z$, define:

$$R \oplus S\colon X \to Z \quad \text{by} \quad xR \oplus Sz \quad \text{if and only if} \quad \forall y \in Y \quad xRy \quad \text{or} \quad ySz$$

- The suspension of a linearly distributive category can be regarded in the obvious manner as a linear bicategory with a single 0-cell.

We'll soon see lots more examples.

Next?

∞-Linear categories

or should that be

Linear ∞-categories