

Presentations and algebraic colimits of enriched monads for a subcategory of arities

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Motivation

- Signatures and presentations for monads and theories (relative to a subcategory of arities) have been previously studied mainly in the context of *locally presentable* enriched categories; see e.g.
 - ▶ [5] G. M. Kelly and A. J. Power, *Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads*, J. Pure Appl. Algebra 89 (1993), no. 1-2, 163–179.
 - ▶ [6] Stephen Lack, *On the monadicity of finitary monads*, J. Pure Appl. Algebra 140 (1999), no. 1, 65–73.
 - ▶ [3] John Bourke and Richard Garner, *Monads and theories*, Adv. Math. 351 (2019), 1024–1071.

Motivation

- Kelly and Power show in [5] that if \mathcal{C} is a locally finitely presentable \mathcal{V} -category over a locally finitely presentable closed category \mathcal{V} , then the forgetful functor $\mathcal{W} : \mathbf{Mnd}_f(\mathcal{C}) \rightarrow \mathbf{End}_f(\mathcal{C})$ has a left adjoint.
- Lack then shows in [6] that the forgetful functor $\mathcal{U} : \mathbf{Mnd}_f(\mathcal{C}) \rightarrow \mathbf{Sig}_f(\mathcal{C})$ is monadic. In particular, any finitary \mathcal{V} -monad on \mathcal{C} then has a *presentation* in terms of abstract operations and equations.
- Bourke and Garner show in [3] that if $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is a small subcategory of arities in a locally presentable \mathcal{V} -category \mathcal{C} over a locally presentable \mathcal{V} , then $\mathcal{U} : \mathbf{Mnd}_{\mathcal{J}\text{Nerv}}(\mathcal{C}) \rightarrow \mathbf{Sig}_{\mathcal{J}}(\mathcal{C})$ is monadic, and $\mathbf{Mnd}_{\mathcal{J}\text{Nerv}}(\mathcal{C})$ has all small (algebraic) colimits.

Motivation

- In this talk, we will discuss a very general framework for studying signatures, presentations, and algebraic colimits of enriched monads relative to a subcategory of arities, which applies even to *locally bounded* enriched categories [10] and the Borceux-Day π -categories [2] (which need not be locally presentable).
- Our results subsume the results of Kelly, Power, and Lack just mentioned, as well as many instances of the Bourke-Garner results just mentioned. Our results also apply in great generality to the Φ -accessible \mathcal{V} -monads studied by Lack and Rosický in
 - ▶ [7] Stephen Lack and Jiří Rosický, *Notions of Lawvere theory*, Appl. Categ. Structures 19 (2011), no. 1, 363–391.

Eleutheric subcategories of arities

- Fix a complete and cocomplete symmetric monoidal closed category \mathcal{V} . A **subcategory of arities** $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a \mathcal{V} -category \mathcal{C} is a small, full, and dense sub- \mathcal{V} -category.
- A subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is **eleutheric** if any \mathcal{V} -functor $F : \mathcal{J} \rightarrow \mathcal{C}$ has a left Kan extension along j , which is moreover preserved by each $\mathcal{C}(J, -) : \mathcal{C} \rightarrow \mathcal{V}$ ($J \in \mathbf{ob} \mathcal{J}$).
- $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is eleutheric iff j presents \mathcal{C} as a free Φ -cocompletion of \mathcal{J} for a class of small weights Φ .

\mathcal{J} -ary \mathcal{V} -monads

- If $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is a subcategory of arities, then a \mathcal{V} -endofunctor $H : \mathcal{C} \rightarrow \mathcal{C}$ is **\mathcal{J} -ary** if it preserves left Kan extensions along j . We let **$\mathbf{End}_{\mathcal{J}}(\mathcal{C})$** be the category of \mathcal{J} -ary \mathcal{V} -endofunctors on \mathcal{C} and **$\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$** the category of \mathcal{J} -ary \mathcal{V} -monads on \mathcal{C} .
- If $j : \mathcal{J} \hookrightarrow \mathcal{C}$ presents \mathcal{C} as a free Φ -cocompletion of \mathcal{J} (and hence is eleutheric), then $H : \mathcal{C} \rightarrow \mathcal{C}$ is \mathcal{J} -ary iff H preserves Φ -colimits.
- If $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is eleutheric, then **\mathcal{V} -CAT** $(\mathcal{J}, \mathcal{C}) \simeq \mathbf{End}_{\mathcal{J}}(\mathcal{C})$ via precomposition with j and left Kan extension along j .
- If $\mathcal{C} = \mathcal{V}$ and $j : \mathcal{J} \hookrightarrow \mathcal{V}$ is an eleutheric *system* of arities (i.e. \mathcal{J} contains I and is closed under \otimes), then **$\mathbf{Mnd}_{\mathcal{J}}(\mathcal{V}) \simeq \mathbf{Th}_{\mathcal{J}}$** , the category of **$\mathcal{J}$ -theories** [9].

Examples of eleutheric subcategories of arities

- The subcategory of arities $j : \mathcal{C}_\alpha \hookrightarrow \mathcal{C}$ of α -presentable objects in a locally α -presentable \mathcal{V} -category \mathcal{C} over a locally α -presentable closed category \mathcal{V} . The \mathcal{J} -ary \mathcal{V} -endofunctors preserve conical α -filtered colimits, and the \mathcal{J} -ary \mathcal{V} -monads correspond to the enriched Lawvere theories of Nishizawa-Power [13].
- The subcategory of arities $j : \mathcal{C}_\Phi \hookrightarrow \mathcal{C}$ consisting of suitable Φ -presentable objects in a locally Φ -presentable \mathcal{V} -category for a suitable class of weights Φ [7]. The \mathcal{J} -ary \mathcal{V} -endofunctors preserve Φ -flat colimits, and the \mathcal{J} -ary (i.e. Φ -**accessible**) \mathcal{V} -monads correspond to the Lawvere Φ -theories of Lack-Rosický [7].
- In particular, the subcategory of arities $j : \mathcal{C}_\mathbb{D} \hookrightarrow \mathcal{C}$ of \mathbb{D} -presentable objects in a locally \mathbb{D} -presentable \mathcal{V} -category over a locally \mathbb{D} -presentable closed category \mathcal{V} for a sound doctrine \mathbb{D} [1], where the \mathcal{J} -ary \mathcal{V} -endofunctors preserve conical \mathbb{D} -filtered colimits.

Examples of eleutheric subcategories of arities

- The subcategory of arities $j : \{1\} \hookrightarrow \mathcal{V}$ consisting of the unit object. The \mathcal{J} -ary \mathcal{V} -endofunctors have the form $X \otimes (-) : \mathcal{V} \rightarrow \mathcal{V}$ ($X \in \mathbf{ob}\mathcal{V}$), and the \mathcal{J} -ary \mathcal{V} -monads correspond to monoids in \mathcal{V} .
- The subcategory of arities $j : \mathbb{N}_{\mathcal{V}} \hookrightarrow \mathcal{V}$ consisting of the finite copowers of the unit object in any π -category \mathcal{V} [2]. The \mathcal{J} -ary \mathcal{V} -endofunctors preserve $\mathbb{N}_{\mathcal{V}}$ -flat colimits (incl. sifted colimits), and the \mathcal{J} -ary \mathcal{V} -monads correspond to the enriched finite power theories of Borceux-Day [2].

Factegories

- We will now discuss the other assumption(s) that we will need to impose on our subcategories of arities.
- \mathcal{V} is a **closed factegory** if \mathcal{V} is equipped with an enriched factorization system $(\mathcal{E}, \mathcal{M})$ [8].
- If \mathcal{V} is a closed factegory, then a **\mathcal{V} -factegory** is a \mathcal{V} -category \mathcal{C} equipped with an enriched factorization system $(\mathcal{E}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}})$ that is **compatible with** $(\mathcal{E}, \mathcal{M})$, i.e. each $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathcal{V}$ ($C \in \mathbf{ob}\mathcal{C}$) preserves the right class.
- The \mathcal{V} -factegory \mathcal{C} is **cocomplete** if \mathcal{C} is cocomplete and has arbitrary cointersections of $\mathcal{E}_{\mathcal{C}}$ -morphisms.

Boundedness

- Let \mathcal{C} and \mathcal{D} be \mathcal{V} -categories with conical α -filtered colimits. A \mathcal{V} -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is **α -bounded** if for any α -filtered diagram $D : \mathcal{A} \rightarrow \mathcal{C}_0$ with colimit $\mathbf{colim} D$ and any \mathcal{M} -cocone $m = (m_A : DA \rightarrow C)_A$, if $\mathbf{colim} D \xrightarrow{\bar{m}} C$ lies in \mathcal{E} , then $\mathbf{colim} FD \xrightarrow{\overline{Fm}} FC$ lies in \mathcal{E} . (In Kelly's terminology [4], we also say that F **preserves the \mathcal{E} -tightness of α -filtered \mathcal{M} -cocones.**)
- We say that $F : \mathcal{C} \rightarrow \mathcal{D}$ is **bounded** if F is α -bounded for some α . If $C \in \mathbf{ob}\mathcal{C}$, then C is **α -bounded** if $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathcal{V}$ is α -bounded.
- If each of $(\mathcal{E}, \mathcal{M}), (\mathcal{E}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}}), (\mathcal{E}_{\mathcal{D}}, \mathcal{M}_{\mathcal{D}})$ is just **(Iso, All)**, then $F : \mathcal{C} \rightarrow \mathcal{D}$ is α -bounded iff F preserves conical α -filtered colimits.

Bounded subcategories of arities

If $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is a subcategory of arities in a \mathcal{V} -category \mathcal{C} , then \mathcal{J} is **(α -)bounded** if every $J \in \mathbf{ob} \mathcal{J}$ is (α -)bounded.

Proposition

If $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is an α -bounded subcategory of arities in a cocomplete \mathcal{V} -category \mathcal{C} and $H : \mathcal{C} \rightarrow \mathcal{C}$ is a left Kan extension along j , then H is α -bounded. In particular, any \mathcal{J} -ary \mathcal{V} -endofunctor $H : \mathcal{C} \rightarrow \mathcal{C}$ is α -bounded.

Blanket assumptions

All of our results hold for any subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a \mathcal{V} -category \mathcal{C} satisfying the following assumptions:

- \mathcal{V} is complete, cocomplete, and a closed factegory.
- \mathcal{C} is a cocomplete \mathcal{V} -factegory that is cotensored, and $(\mathcal{E}, \mathcal{M})$ is **proper** or \mathcal{C} is \mathcal{E} -cowellpowered.
- $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is bounded and eleutheric.

Aside from the Lack-Rosický example, **all** of the previous examples satisfy these blanket assumptions. If \mathcal{V} is *locally bounded* and \mathcal{E} -cowellpowered, then the Lack-Rosický example **does** satisfy these blanket assumptions (specifically, the canonical subcategory of arities is **bounded**).

Algebraically free \mathcal{J} -ary \mathcal{V} -monads

- A \mathcal{V} -monad \mathbb{T} on \mathcal{C} is an **algebraically free \mathcal{V} -monad** on a \mathcal{V} -endofunctor $H : \mathcal{C} \rightarrow \mathcal{C}$ if $U^{\mathbb{T}} : \mathbb{T}\text{-Alg} \rightarrow \mathcal{C}$ is isomorphic to $U^H : H\text{-Alg} \rightarrow \mathcal{C}$ in $\mathcal{V}\text{-CAT}/\mathcal{C}$ (cf. [4]).
- Kelly showed [4] that $H : \mathcal{C} \rightarrow \mathcal{C}$ has an algebraically free \mathcal{V} -monad \mathbb{T}_H iff $U^H : H\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint, in which case \mathbb{T}_H is the induced \mathcal{V} -monad.

Theorem

If $H : \mathcal{C} \rightarrow \mathcal{C}$ is \mathcal{J} -ary, then $U^H : H\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint, and the resulting algebraically free \mathcal{V} -monad \mathbb{T}_H on H is \mathcal{J} -ary. So $W : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow \mathbf{End}_{\mathcal{J}}(\mathcal{C})$ is monadic.

Free \mathcal{J} -ary \mathcal{V} -monads on \mathcal{J} -signatures

- A \mathcal{J} -signature in \mathcal{C} is a functor $\Sigma : \mathbf{ob} \mathcal{J} \rightarrow \mathcal{C}_0$. The category $\mathbf{Sig}_{\mathcal{J}}(\mathcal{C})$ of \mathcal{J} -signatures is then the functor category $\mathbf{CAT}(\mathbf{ob} \mathcal{J}, \mathcal{C}_0)$.
- A Σ -algebra in \mathcal{C} is an object $C \in \mathbf{ob} \mathcal{C}$ equipped with morphisms $\Sigma J \rightarrow [\mathcal{C}(J, C), C]$ ($J \in \mathbf{ob} \mathcal{J}$), yielding a \mathcal{V} -category $U^{\Sigma} : \Sigma\text{-Alg} \rightarrow \mathcal{C}$ over \mathcal{C} .

Theorem

Every \mathcal{J} -signature Σ has a free \mathcal{J} -ary \mathcal{V} -monad \mathbb{T}_{Σ} with $\mathbb{T}_{\Sigma}\text{-Alg} \cong \Sigma\text{-Alg}$ in $\mathcal{V}\text{-CAT}/\mathcal{C}$. So $U : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow \mathbf{Sig}_{\mathcal{J}}(\mathcal{C})$ has a left adjoint, and is even monadic.

Algebraic colimits of \mathcal{V} -monads

- An **algebraic colimit** of a small diagram of \mathcal{V} -monads $\mathbb{M} : \mathcal{K} \rightarrow \mathbf{Mnd}(\mathcal{C})$ is a colimit \mathbb{T} that is sent to a limit by the semantics functor $\mathbf{Alg} : \mathbf{Mnd}(\mathcal{C})^{\text{op}} \rightarrow \mathcal{V}\text{-CAT}/\mathcal{C}$ (cf. [4]).
- Let $U^{\mathbb{M}} : \mathbb{M}\text{-Alg} \rightarrow \mathcal{C}$ be the limit of

$$\mathcal{K}^{\text{op}} \xrightarrow{\mathbb{M}^{\text{op}}} \mathbf{Mnd}(\mathcal{C})^{\text{op}} \xrightarrow{\mathbf{Alg}} \mathcal{V}\text{-CAT}/\mathcal{C}.$$

Kelly showed [4] that \mathbb{M} has an algebraic colimit $\mathbb{T}_{\mathbb{M}}$ iff $U^{\mathbb{M}} : \mathbb{M}\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint, in which case $\mathbb{T}_{\mathbb{M}}$ is the induced \mathcal{V} -monad.

Algebraic colimits of \mathcal{J} -ary \mathcal{V} -monads

Theorem

Let $\mathbb{M} : \mathcal{K} \rightarrow \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ be a small diagram. Then $U^{\mathbb{M}} : \mathbb{M}\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint, and the resulting algebraic colimit $\mathbb{T}_{\mathbb{M}}$ is \mathcal{J} -ary. So $\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ has small algebraic colimits.

Presentations of \mathcal{J} -ary \mathcal{V} -monads

- A **\mathcal{J} -presentation** is a parallel pair of \mathcal{J} -signature morphisms $\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma$. One can define the full sub- \mathcal{V} -category $P\text{-Alg} \hookrightarrow \Sigma\text{-Alg}$ of the **P -algebras**.
- A \mathcal{J} -presentation $(\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma)$ **presents** a \mathcal{J} -ary \mathcal{V} -monad \mathbb{T} if \mathbb{T} is an algebraic coequalizer of the induced parallel pair $\bar{\alpha}, \bar{\beta} : \mathbb{T}_\Gamma \rightrightarrows \mathbb{T}_\Sigma$, i.e. there is a coequalizer $\mathbb{T}_\Gamma \rightrightarrows \mathbb{T}_\Sigma \rightarrow \mathbb{T}$ in $\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ preserved by the semantics functor $\mathbf{Alg} : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow (\mathcal{V}\text{-CAT}/\mathcal{C})^{\text{op}}$.

Theorem

Every \mathcal{J} -ary \mathcal{V} -monad has a (canonical) \mathcal{J} -presentation, and every \mathcal{J} -presentation P presents a \mathcal{J} -ary \mathcal{V} -monad \mathbb{T}_P with $\mathbb{T}_P\text{-Alg} \cong P\text{-Alg}$ in $\mathcal{V}\text{-CAT}/\mathcal{C}$.

Locally bounded \mathcal{V} -categories

Theorem

Let \mathcal{C} be a locally bounded \mathcal{V} -category over a locally bounded closed category \mathcal{V} [10]. Then any subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is bounded.

- Thus, all of our results hold for any eleutheric subcategory of arities in any locally bounded \mathcal{V} -category over a locally bounded closed category \mathcal{V} .
- In particular, if \mathcal{V} is locally bounded and \mathcal{E} -cowellpowered and Φ is a class of weights studied by Lack-Rosický [7], then any locally Φ -presentable \mathcal{V} -category \mathcal{C} is locally bounded, and hence its canonical subcategory of arities $j : \mathcal{C}_\Phi \hookrightarrow \mathcal{C}$ is bounded (and eleutheric).

In summary...

- We have developed a very general framework for studying presentations and algebraic colimits for enriched monads relative to a subcategory of arities \mathcal{J} , which includes locally bounded enriched categories and Borceux-Day π -categories.
- Specifically, we have seen that free \mathcal{J} -ary \mathcal{V} -monads on \mathcal{J} -ary \mathcal{V} -endofunctors and \mathcal{J} -signatures, algebraic colimits of \mathcal{J} -ary \mathcal{V} -monads, and presentations of \mathcal{J} -ary \mathcal{V} -monads can be obtained for any bounded and eleutheric subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a \mathcal{V} -category \mathcal{C} satisfying mild assumptions.

In summary...

- In particular, our results apply to any eleutheric subcategory of arities in any locally bounded \mathcal{V} -category over a locally bounded closed category \mathcal{V} .
- The preprint [12] will (hopefully!) be out in the next few weeks. We have also developed a very “user-friendly” reformulation of the notion of \mathcal{J} -presentation that will appear (with many examples) in forthcoming work [11].

Thank you!

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